

Schrödinger-Maxwell systems on Hadamard manifolds

Csaba Farkas^{*,1,3} Alexandru Kristály^{**,2,3}

¹Department of Mathematics and Informatics, Sapiientia University, Tg. Mureş, Romania

² Department of Economics, Babeş-Bolyai University, Cluj-Napoca, Romania

³Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary

February 12, 2016

Abstract. In this paper we study nonlinear Schrödinger-Maxwell systems on n -dimensional Hadamard manifolds, $3 \leq n \leq 5$. The main difficulty resides in the lack of compactness of such manifolds which is recovered by exploring suitable isometric actions. By combining variational arguments, some existence, uniqueness and multiplicity of isometry-invariant weak solutions are established for the Schrödinger-Maxwell system depending on the behavior of the nonlinear term.

1. INTRODUCTION AND MAIN RESULTS

1.1. Motivation. The Schrödinger-Maxwell system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi eu^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

describes the statical behavior of a charged non-relativistic quantum mechanical particle interacting with the electromagnetic field. More precisely, the unknown terms $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ are the fields associated to the particle and the electric potential, respectively. Here and in the sequel, the quantities m , e , ω and \hbar are the mass, charge, phase, and Planck's constant, respectively, while $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function verifying some growth conditions. In fact, system (1.1) comes from the evolutionary nonlinear Schrödinger equation by using a Lyapunov-Schmidt reduction.

The Schrödinger-Maxwell system (or its variants) has been the object of various investigations in the last two decades. Without sake of completeness, we recall in the sequel some important contributions to the study of system (1.1). Benci and Fortunato [4] considered the case of $f(x, s) = |s|^{p-2}s$ with $p \in (4, 6)$ by proving the existence of infinitely many radial solutions for (1.1); their main step relies on the reduction of system (1.1) to the investigation of critical points of a "one-variable" energy functional associated with (1.1). Based on the idea of Benci and Fortunato, under various growth assumptions on f further existence/multiplicity results can be found in Ambrosetti and Ruiz [1], Azzolini [2], Azzollini, d'Avenia and Pomponio [3], d'Avenia [10], d'Aprile and Mugnai [8], Cerami and Vaira [7], Kristály and Repovš [21], Ruiz [24], Sun, Chen and Nieto [26], Wang and Zhou [31], Zhao and Zhao [35], and references therein. By means of a Pohozaev-type identity, d'Aprile and Mugnai [9] proved the non-existence of non-trivial solutions to system (1.1) whenever $f \equiv 0$ or $f(x, s) = |s|^{p-2}s$ and $p \in (0, 2] \cup [6, \infty)$.

In recent years considerable efforts have been done to describe various nonlinear phenomena in *curves spaces* (which are mainly understood in linear structures), e.g. optimal mass transportation on metric measure spaces, geometric functional inequalities and optimization problems on Riemannian/Finsler manifolds, etc. In particular, this research stream reached as well

*Email address of Cs. Farkas: farkas.csaba2008@gmail.com

**Email address of A. Kristály: alexandrukristaly@yahoo.com

the study of Schrödinger-Maxwell systems. Indeed, in the last five years Schrödinger-Maxwell systems has been studied on n -dimensional *compact Riemannian manifolds* ($2 \leq n \leq 5$) by Druet and Hebey [11], Hebey and Wei [15], Ghimenti and Micheletti [12, 13] and Thizy [29, 30]. More precisely, in the aforementioned papers various forms of the system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(u) & \text{in } M, \\ -\Delta_g \phi + \phi = 4\pi eu^2 & \text{in } M, \end{cases} \quad (1.2)$$

has been considered, where (M, g) is a compact Riemannian manifold and Δ_g is the Laplace-Beltrami operator, by proving existence results with further qualitative property of the solution(s). As expected, the compactness of (M, g) played a crucial role in these investigations.

As far as we know, no result is available in the literature concerning Maxwell-Schrödinger systems on *non-compact Riemannian manifolds*. Motivated by this fact, the purpose of the present paper is to provide existence, uniqueness and multiplicity results in the case of the Maxwell-Schrödinger system in such a non-compact setting. Since this problem is very general, we shall restrict our study to *Hadamard manifolds* (simply connected, complete Riemannian manifolds with non-positive sectional curvature). Although any Hadamard manifold (M, g) is diffeomorphic to \mathbb{R}^n , $n = \dim M$ (cf. Cartan's theorem), this is a wide class of non-compact Riemannian manifold including important geometric objects (as Euclidean spaces, hyperbolic spaces, the space of symmetric positive definite matrices endowed with a suitable Killing metric), see Bridson and Haefliger [6].

To be more precise, we shall consider the Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\phi = \lambda \alpha(x)f(u) & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \quad (\mathcal{SM}_\lambda)$$

where (M, g) is an n -dimensional Hadamard manifold ($3 \leq n \leq 5$), $e, q > 0$ are positive numbers, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\alpha : M \rightarrow \mathbb{R}$ is a measurable function, and $\lambda > 0$ is a parameter. The solutions (u, ϕ) of (\mathcal{SM}_λ) are sought in the Sobolev space $H_g^1(M) \times H_g^1(M)$. In order to handle the lack of compactness of (M, g) , a Lions-type symmetrization argument will be used, based on the action of a suitable subgroup of the group of isometries of (M, g) . More precisely, we shall adapt the main results of Skrzypczak and Tintarev [27] to our setting concerning Sobolev spaces in the presence of group-symmetries. By exploring variational arguments (principle of symmetric criticality, minimization and mountain pass arguments), we consider the following problems describing roughly as well the main achievements:

- A. *Schrödinger-Maxwell systems of Poisson type*: $\lambda = 1$ and $f \equiv 1$. We prove the existence of the unique weak solution $(u, \phi) \in H_g^1(M) \times H_g^1(M)$ to (\mathcal{SM}_1) while if α has some radial property (formulated in terms of the isometry group), the unique weak solution is isometry-invariant, see Theorem 1.1. Moreover, we prove a rigidity result which states that a specific profile function uniquely determines the structure of the Hadamard manifold (M, g) , see Theorem 1.2.
- B. *Schrödinger-Maxwell systems involving sublinear terms at infinity*: f is sublinear at infinity. We prove that for small values of $\lambda > 0$ system (\mathcal{SM}_λ) has only the trivial solution, while for enough large $\lambda > 0$ the system (\mathcal{SM}_λ) has at least two distinct, non-zero, isometry-invariant weak solutions, see Theorem 1.3.
- C. *Schrödinger-Maxwell systems involving oscillatory terms*: f oscillates near the origin. We prove that system (\mathcal{SM}_1) has infinitely many distinct, non-zero, isometry-invariant weak solutions which converge to 0 in the $H_g^1(M)$ -norm, see Theorem 1.4.

In the sequel, we shall formulate rigorously our main results with some comments.

1.2. Statement of main results. Let (M, g) be an n -dimensional Hadamard manifold, $3 \leq n \leq 6$. The pair $(u, \phi) \in H_g^1(M) \times H_g^1(M)$ is a *weak solution* to the system (\mathcal{SM}_λ) if

$$\int_M (\langle \nabla_g u, \nabla_g v \rangle + uv + eu\phi v) dv_g = \lambda \int_M \alpha(x) f(u) v dv_g \text{ for all } v \in H_g^1(M), \quad (1.3)$$

$$\int_M (\langle \nabla_g \phi, \nabla_g \psi \rangle + \phi\psi) dv_g = q \int_M u^2 \psi dv_g \text{ for all } \psi \in H_g^1(M). \quad (1.4)$$

For later use, we denote by $\text{Isom}_g(M)$ the group of isometries of (M, g) and let G be a subgroup of $\text{Isom}_g(M)$. A function $u : M \rightarrow \mathbb{R}$ is G -invariant if $u(\sigma(x)) = u(x)$ for every $x \in M$ and $\sigma \in G$. Furthermore, $u : M \rightarrow \mathbb{R}$ is *radially symmetric w.r.t. $x_0 \in M$* if u depends on $d_g(x_0, \cdot)$, d_g being the Riemannian distance function. The *fixed point set of G* on M is given by $\text{Fix}_M(G) = \{x \in M : \sigma(x) = x \text{ for all } \sigma \in G\}$. For a given $x_0 \in M$, we introduce the following hypothesis which will be crucial in our investigations:

$(H_G^{x_0})$ *The group G is a compact connected subgroup of $\text{Isom}_g(M)$ such that $\text{Fix}_M(G) = \{x_0\}$.*

Remark 1.1. In the sequel, we provide some concrete Hadamard manifolds and group of isometries for which hypothesis **$(H_G^{x_0})$** is satisfied:

- *Euclidean spaces.* If $(M, g) = (\mathbb{R}^n, g_{\text{euc}})$ is the usual Euclidean space, then $x_0 = 0$ and $G = SO(n_1) \times \dots \times SO(n_l)$ with $n_j \geq 2$, $j = 1, \dots, l$ and $n_1 + \dots + n_l = n$, satisfy **$(H_G^{x_0})$** , where $SO(k)$ is the special orthogonal group in dimension k . Indeed, we have $\text{Fix}_{\mathbb{R}^n}(G) = \{0\}$.
- *Hyperbolic spaces.* Let us consider the Poincaré ball model $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the Riemannian metric $g_{\text{hyp}}(x) = (g_{ij}(x))_{i,j=1,\dots,n} = \frac{4}{(1-|x|^2)^2} \delta_{ij}$. It is well known that $(\mathbb{H}^n, g_{\text{hyp}})$ is a homogeneous Hadamard manifold with constant sectional curvature -1 . Hypothesis **$(H_G^{x_0})$** is verified with the same choices as above.
- *Symmetric positive definite matrices.* Let $\text{Sym}(n, \mathbb{R})$ be the set of symmetric $n \times n$ matrices with real values, $P(n, \mathbb{R}) \subset \text{Sym}(n, \mathbb{R})$ be the cone of symmetric positive definite matrices, and $P(n, \mathbb{R})_1$ be the subspace of matrices in $P(n, \mathbb{R})$ with determinant one. The set $P(n, \mathbb{R})$ is endowed with the scalar product

$$\langle U, V \rangle_X = \text{Tr}(X^{-1} V X^{-1} U) \text{ for all } X \in P(n, \mathbb{R}), U, V \in T_X(P(n, \mathbb{R})) \simeq \text{Sym}(n, \mathbb{R}),$$

where $\text{Tr}(Y)$ denotes the trace of $Y \in \text{Sym}(n, \mathbb{R})$. One can prove that $(P(n, \mathbb{R})_1, \langle \cdot, \cdot \rangle)$ is a homogeneous Hadamard manifold (with non-constant sectional curvature) and the special linear group $SL(n)$ leaves $P(n, \mathbb{R})_1$ invariant and acts transitively on it. Moreover, for every $\sigma \in SL(n)$, the map $[\sigma] : P(n, \mathbb{R})_1 \rightarrow P(n, \mathbb{R})_1$ defined by $[\sigma](X) = \sigma X \sigma^t$, is an isometry, where σ^t denotes the transpose of σ . If $G = SO(n)$, we can prove that $\text{Fix}_{P(n, \mathbb{R})_1}(G) = \{I_n\}$, where I_n is the identity matrix; for more details, see Kristály [18].

For $x_0 \in M$ fixed, we also introduce the hypothesis

(α^{x_0}) *The function $\alpha : M \rightarrow \mathbb{R}$ is non-zero, non-negative and radially symmetric w.r.t. x_0 .*

Our results are divided into three classes:

A. Schrödinger-Maxwell systems of Poisson type. Dealing with a Poisson-type system, we set $\lambda = 1$ and $f \equiv 1$ in (\mathcal{SM}_λ) . For abbreviation, we simply denote (\mathcal{SM}_1) by (\mathcal{SM}) .

Theorem 1.1. *Let (M, g) be an n -dimensional homogeneous Hadamard manifold ($3 \leq n \leq 6$), and $\alpha \in L^2(M)$ be a non-negative function. Then there exists a unique, non-negative weak solution $(u_0, \phi_0) \in H_g^1(M) \times H_g^1(M)$ to problem (\mathcal{SM}) . Moreover, if $x_0 \in M$ is fixed and α satisfies (α^{x_0}) , then (u_0, ϕ_0) is G -invariant w.r.t. any group $G \subset \text{Isom}_g(M)$ which satisfies $(H_G^{x_0})$.*

Remark 1.2. Let (M, g) be either the n -dimensional Euclidean space $(\mathbb{R}^n, g_{\text{euc}})$ or hyperbolic space $(\mathbb{H}^n, g_{\text{hyp}})$, and fix $G = SO(n_1) \times \dots \times SO(n_l)$ for a splitting of $n = n_1 + \dots + n_l$ with $n_j \geq 2$, $j = 1, \dots, l$. If α is radially symmetric (w.r.t. $x_0 = 0$), Theorem 1.1 states that the unique solution (u_0, ϕ_0) to the Poisson-type Schrödinger-Maxwell system (\mathcal{SM}) is not only invariant w.r.t. the group G but also with any compact connected subgroup \tilde{G} of $\text{Isom}_g(M)$ with the same fixed point property $\text{Fix}_M(\tilde{G}) = \{0\}$; thus, in particular, (u_0, ϕ_0) is invariant w.r.t. the whole group $SO(n)$, i.e. (u_0, ϕ_0) is radially symmetric.

For every $c \leq 0$, let $\mathbf{s}_c, \mathbf{ct}_c : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\mathbf{s}_c(r) = \begin{cases} r & \text{if } c = 0, \\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & \text{if } c < 0, \end{cases} \quad \text{and} \quad \mathbf{ct}_c(r) = \begin{cases} \frac{1}{r} & \text{if } c = 0, \\ \sqrt{-c} \coth(\sqrt{-c}r) & \text{if } c < 0. \end{cases} \quad (1.5)$$

For $c \leq 0$ and $3 \leq n \leq 6$ we consider the ordinary differential equations system

$$\begin{cases} -h_1''(r) - (n-1)\mathbf{ct}_c(r)h_1'(r) + h_1(r) + eh_1(r)h_2(r) - \alpha_0(r) = 0, & r \geq 0; \\ -h_2''(r) - (n-1)\mathbf{ct}_c(r)h_2'(r) + h_2(r) - qh_1(r)^2 = 0, & r \geq 0; \\ \int_0^\infty (h_1'(r)^2 + h_1^2(r))\mathbf{s}_c(r)^{n-1}dr < \infty; \\ \int_0^\infty (h_2'(r)^2 + h_2^2(r))\mathbf{s}_c(r)^{n-1}dr < \infty, \end{cases} \quad (\mathcal{R})$$

where $\alpha_0 : [0, \infty) \rightarrow [0, \infty)$ satisfies the integrability condition $\alpha_0 \in L^2([0, \infty), \mathbf{s}_c(r)^{n-1}dr)$.

We shall show (see Lemma 3.2) that (\mathcal{R}) has a unique, non-negative solution $(h_1^c, h_2^c) \in C^\infty(0, \infty) \times C^\infty(0, \infty)$. In fact, the following rigidity result can be stated:

Theorem 1.2. *Let (M, g) be an n -dimensional homogeneous Hadamard manifold ($3 \leq n \leq 6$) with sectional curvature $\mathbf{K} \leq c \leq 0$. Let $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^2(M)$ be such that hypotheses $(H_G^{x_0})$ and (α^{x_0}) are satisfied. If $\alpha^{-1}(t) \subset M$ has null Riemannian measure for every $t \geq 0$, then the following statements are equivalent:*

- (i) $(h_1^c(d_g(x_0, \cdot)), h_2^c(d_g(x_0, \cdot)))$ is the unique pointwise solution of (\mathcal{SM}) ;
- (ii) (M, g) is isometric to the space form with constant sectional curvature $\mathbf{K} = c$.

B. Schrödinger-Maxwell systems involving sublinear terms at infinity. In this part, we focus our attention to Schrödinger-Maxwell systems involving sublinear nonlinearities. To state our result we consider a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ which verifies the following assumptions:

- (f₁) $\frac{f(s)}{s} \rightarrow 0$ as $s \rightarrow 0^+$;
- (f₂) $\frac{f(s)}{s} \rightarrow 0$ as $s \rightarrow \infty$;
- (f₃) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \int_0^s f(t)dt$, $s \geq 0$.

Remark 1.3. (a) Due to (f_1) , it is clear that $f(0) = 0$, thus we can extend continuously the function $f : [0, \infty) \rightarrow \mathbb{R}$ to the whole \mathbb{R} by $f(s) = 0$ for $s \leq 0$; thus, $F(s) = 0$ for $s \leq 0$.

(b) (f_1) and (f_2) mean that f is superlinear at the origin and sublinear at infinity, respectively. The function $f(s) = \ln(1 + s^2)$, $s \geq 0$, verifies hypotheses $(f_1) - (f_3)$.

Theorem 1.3. *Let (M, g) be an n -dimensional homogeneous Hadamard manifold ($3 \leq n \leq 5$), $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(H_G^{x_0})$ and (α^{x_0}) are satisfied. If the continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ satisfies assumptions $(f_1) - (f_3)$, then*

- (i) *there exists $\tilde{\lambda}_0 > 0$ such that if $0 < \lambda < \tilde{\lambda}_0$, system (\mathcal{SM}_λ) has only the trivial solution;*
- (ii) *there exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$, system (\mathcal{SM}_λ) has at least two distinct non-zero, non-negative G -invariant weak solutions in $H_g^1(M) \times H_g^1(M)$.*

Remark 1.4. (a) By a three critical points result of Ricceri [23] one can prove that the number of solutions for system (\mathcal{SM}_λ) is stable under small nonlinear perturbations $g : [0, \infty) \rightarrow \mathbb{R}$ of subcritical type, i.e., $g(s) = o(|s|^{2^*-1})$ as $s \rightarrow \infty$, $2^* = \frac{2n}{n-2}$, whenever $\lambda > \lambda_0$.

(b) Working with sublinear nonlinearities, Theorem 1.3 complements several results where f has a superlinear growth at infinity, e.g., $f(s) = |s|^{p-2}s$ with $p \in (4, 6)$.

C. Schrödinger-Maxwell systems involving oscillatory terms. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function with $F(s) = \int_0^s f(t)dt$. We assume:

$$(f_0^1) \quad -\infty < \liminf_{s \rightarrow 0} \frac{F(s)}{s^2} \leq \limsup_{s \rightarrow 0} \frac{F(s)}{s^2} = +\infty;$$

(f_0^2) there exists a sequence $\{s_j\}_j \subset (0, 1)$ converging to 0 such that $f(s_j) < 0$, $j \in \mathbb{N}$.

Theorem 1.4. *Let (M, g) be an n -dimensional homogeneous Hadamard manifold ($3 \leq n \leq 5$), $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(H_G^{x_0})$ and (α^{x_0}) are satisfied. If $f : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function satisfying (f_0^1) and (f_0^2) , then there exists a sequence $\{(u_j^0, \phi_{u_j^0})\}_j \subset H_g^1(M) \times H_g^1(M)$ of distinct, non-negative G -invariant weak solutions to (\mathcal{SM}) such that*

$$\lim_{j \rightarrow \infty} \|u_j^0\|_{H_g^1(M)} = \lim_{j \rightarrow \infty} \|\phi_{u_j^0}\|_{H_g^1(M)} = 0.$$

Remark 1.5. (a) (f_0^1) and (f_0^2) imply $f(0) = 0$; thus we can extend f as in Remark 1.3 (a).

(b) Under the assumptions of Theorem 1.4 we consider the perturbed Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\phi = \lambda\alpha(x)[f(u) + \varepsilon g(u)] & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \quad (\mathcal{SM}_\varepsilon)$$

where $\varepsilon > 0$ and $g : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function with $g(0) = 0$. Arguing as in the proof of Theorem 1.4, a careful energy control provides the following statement: for every $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that $(\mathcal{SM}_\varepsilon)$ has at least k distinct, G -invariant weak solutions $(u_{j,\varepsilon}, \phi_{u_{j,\varepsilon}})$, $j \in \{1, \dots, k\}$, whenever $\varepsilon \in [-\varepsilon_k, \varepsilon_k]$. Moreover, one can prove that $\|u_{j,\varepsilon}\|_{H_g^1(M)} < \frac{1}{j}$ and $\|\phi_{u_{j,\varepsilon}}\|_{H_g^1(M)} < \frac{1}{j}$, $j \in \{1, \dots, k\}$. Note that a similar phenomenon has been described for Dirichlet problems in Kristály and Moroşanu [19].

(c) Theorem 1.4 complements some results from the literature where $f : \mathbb{R} \rightarrow \mathbb{R}$ has the symmetry property $f(s) = -f(-s)$ for every $s \in \mathbb{R}$ and verifies an Ambrosetti-Rabinowitz-type assumption. Indeed, in such cases, the symmetric version of the mountain pass theorem provides a sequence of weak solutions for the studied Schrödinger-Maxwell system.

2. PRELIMINARIES

2.1. Elements from Riemannian geometry. In the sequel, let $n \geq 3$ and (M, g) be an n -dimensional Hadamard manifold (i.e., (M, g) is a complete, simply connected Riemannian manifold with nonpositive sectional curvature). Let $T_x M$ be the tangent space at $x \in M$, $TM = \bigcup_{x \in M} T_x M$ be the tangent bundle, and $d_g : M \times M \rightarrow [0, +\infty)$ be the distance function associated to the Riemannian metric g . Let $B_g(x, \rho) = \{y \in M : d_g(x, y) < \rho\}$ be the open metric ball with center x and radius $\rho > 0$. If dv_g is the canonical volume element on (M, g) , the volume of a bounded open set $S \subset M$ is $\text{Vol}_g(S) = \int_S dv_g$. If $d\sigma_g$ denotes the $(n-1)$ -dimensional Riemannian measure induced on ∂S by g , $\text{Area}_g(\partial S) = \int_{\partial S} d\sigma_g$ denotes the area of ∂S with respect to the metric g .

Let $p > 1$. The norm of $L^p(M)$ is given by

$$\|u\|_{L^p(M)} = \left(\int_M |u|^p dv_g \right)^{1/p}.$$

Let $u : M \rightarrow \mathbb{R}$ be a function of class C^1 . If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij} u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0, \cdot)| = 1 \quad \text{on } M \setminus \{x_0\}. \quad (2.1)$$

The Laplace-Beltrami operator is given by $\Delta_g u = \text{div}(\nabla_g u)$ whose expression in a local chart of associated coordinates (x^i) is

$$\Delta_g u = g^{ij} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k} \right),$$

where Γ_{ij}^k are the coefficients of the Levi-Civita connection. For enough regular $f : [0, \infty) \rightarrow \mathbb{R}$ one has the formula

$$-\Delta_g(f(d_g(x_0, x))) = -f''(d_g(x_0, x)) - f'(d_g(x_0, x))\Delta_g(d_g(x_0, x)) \quad \text{for a.e. } x \in M. \quad (2.2)$$

When no confusion arises, if $X, Y \in T_x M$, we simply write $|X|$ and $\langle X, Y \rangle$ instead of the norm $|X|_x$ and inner product $g_x(X, Y) = \langle X, Y \rangle_x$, respectively. The $L^p(M)$ norm of $\nabla_g u(x) \in T_x M$ is given by

$$\|\nabla_g u\|_{L^p(M)} = \left(\int_M |\nabla_g u|^p dv_g \right)^{\frac{1}{p}}.$$

The space $H_g^1(M)$ is the completion of $C_0^\infty(M)$ w.r.t. the norm

$$\|u\|_{H_g^1(M)} = \sqrt{\|u\|_{L^2(M)}^2 + \|\nabla_g u\|_{L^2(M)}^2}.$$

Since (M, g) is an n -dimensional Hadamard manifold ($n \geq 3$), according to Hoffman and Spruck [16], the embedding $H_g^1(M) \hookrightarrow L^p(M)$ is continuous for every $p \in [2, 2^*]$, where $2^* = \frac{2n}{n-2}$; see also Hebey [14]. Note that the embedding $H_g^1(M) \hookrightarrow L^p(M)$ is not compact for any $p \in [2, 2^*]$.

For any $c \leq 0$, let $V_{c,n}(\rho) = n\omega_n \int_0^\rho \mathbf{s}_c(t)^{n-1} dt$ be the volume of the ball with radius $\rho > 0$ in the n -dimensional space form (i.e., either the hyperbolic space with sectional curvature c when $c < 0$ or the Euclidean space when $c = 0$), where \mathbf{s}_c is from (1.5) and ω_n is the volume of the unit n -dimensional Euclidean ball. Note that for every $x \in M$, we have

$$\lim_{\rho \rightarrow 0^+} \frac{\text{Vol}_g(B_g(x, \rho))}{V_{c,n}(\rho)} = 1. \quad (2.3)$$

The notation $\mathbf{K} \leq c$ means that the sectional curvature is bounded from above by c at any point and direction.

Let (M, g) be an n -dimensional Hadamard manifold with sectional curvature $\mathbf{K} \leq c \leq 0$. Then we have (see Shen [25] and Wu and Xin [33, Theorems 6.1 & 6.3]):

- *Bishop-Gromov volume comparison theorem*: the function $\rho \mapsto \frac{\text{Vol}_g(B_g(x, \rho))}{V_{c,n}(\rho)}$, $\rho > 0$, is non-decreasing for every $x \in M$. In particular, from (2.3) we have

$$\text{Vol}_g(B_g(x, \rho)) \geq V_{c,n}(\rho) \text{ for all } \rho > 0. \quad (2.4)$$

Moreover, if equality holds in (2.4) for all $x \in M$ and $\rho > 0$ then $\mathbf{K} = c$.

- *Laplace comparison theorem*: $\Delta_g d_g(x_0, x) \geq (n-1)\text{ct}_c(d_g(x_0, x))$ for every $x \in M \setminus \{x_0\}$. If $\mathbf{K} = c$ then we have equality in the latter relation.

2.2. Variational framework. Let (M, g) be an n -dimensional Hadamard manifold, $3 \leq n \leq 6$. We define the energy functional $\mathcal{J}_\lambda : H_g^1(M) \times H_g^1(M) \rightarrow \mathbb{R}$ associated with system (\mathcal{SM}_λ) , namely,

$$\mathcal{J}_\lambda(u, \phi) = \frac{1}{2} \|u\|_{H_g^1(M)}^2 + \frac{e}{2} \int_M \phi u^2 dv_g - \frac{e}{4q} \int_M |\nabla_g \phi|^2 dv_g - \frac{e}{4q} \int_M \phi^2 dv_g - \lambda \int_M \alpha(x) F(u) dv_g.$$

In all our cases (see problems **A**, **B** and **C** above), the functional \mathcal{J}_λ is well-defined and of class C^1 on $H_g^1(M) \times H_g^1(M)$. To see this, we have to consider the second and fifth terms from \mathcal{J}_λ ; the other terms trivially verify the required properties. First, a comparison principle and suitable Sobolev embeddings give that there exists $C > 0$ such that for every $(u, \phi) \in H_g^1(M) \times H_g^1(M)$,

$$0 \leq \int_M \phi u^2 dv_g \leq \left(\int_M \phi^{2^*} dv_g \right)^{\frac{1}{2^*}} \left(\int_M |u|^{\frac{4n}{n+2}} dv_g \right)^{1-\frac{1}{2^*}} \leq C \|\phi\|_{H_g^1(M)} \|u\|_{H_g^1(M)}^2 < \infty,$$

where we used $3 \leq n \leq 6$. If $\mathcal{F} : H_g^1(M) \rightarrow \mathbb{R}$ is the functional defined by $\mathcal{F}(u) = \int_M \alpha(x) F(u) dv_g$, we have:

- Problem **A**: $\alpha \in L^2(M)$ and $F(s) = s$, $s \in \mathbb{R}$, thus $|\mathcal{F}(u)| \leq \|\alpha\|_{L^2(M)} \|u\|_{L^2(M)} < +\infty$ for all $u \in H_g^1(M)$.
- Problems **B** and **C**: the assumptions allow to consider generically that f is subcritical, i.e., there exist $c > 0$ and $p \in [2, 2^*)$ such that $|f(s)| \leq c(|s| + |s|^{p-1})$ for every $s \in \mathbb{R}$. Since $\alpha \in L^\infty(M)$ in every case, we have that $|\mathcal{F}(u)| < +\infty$ for every $u \in H_g^1(M)$ and \mathcal{F} is of class C^1 on $H_g^1(M)$.

STEP 1. The pair $(u, \phi) \in H_g^1(M) \times H_g^1(M)$ is a weak solution of (\mathcal{SM}_λ) if and only if (u, ϕ) is a critical point of \mathcal{J}_λ . Indeed, due to relations (1.3) and (1.4), the claim follows.

By exploring an idea of Benci and Fortunato [4], due to the Lax-Milgram theorem (see e.g. Brezis [5, Corollary 5.8]), we introduce the map $\phi_u : H_g^1(M) \rightarrow H_g^1(M)$ by associating to every $u \in H_g^1(M)$ the unique solution $\phi = \phi_u$ of the Maxwell equation

$$-\Delta_g \phi + \phi = qu^2.$$

We recall some important properties of the function $u \mapsto \phi_u$ which are straightforward adaptations of [21, Proposition 2.1] and [24, Lemma 2.1] to the Riemannian setting:

$$\|\phi_u\|_{H_g^1(M)}^2 = q \int_M \phi_u u^2 dv_g, \quad \phi_u \geq 0; \quad (2.5)$$

$$u \mapsto \int_M \phi_u u^2 dv_g \text{ is convex}; \quad (2.6)$$

$$\int_M (u\phi_u - v\phi_v)(u - v) dv_g \geq 0 \text{ for all } u, v \in H_g^1(M). \quad (2.7)$$

The "one-variable" energy functional $\mathcal{E}_\lambda : H_g^1(M) \rightarrow \mathbb{R}$ associated with system (\mathcal{SM}_λ) is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2} \|u\|_{H_g^1(M)}^2 + \frac{e}{4} \int_M \phi_u u^2 dv_g - \lambda \mathcal{F}(u). \quad (2.8)$$

By using standard variational arguments, one has:

STEP 2. *The pair $(u, \phi) \in H_g^1(M) \times H_g^1(M)$ is a critical point of \mathcal{J}_λ if and only if u is a critical point of \mathcal{E}_λ and $\phi = \phi_u$.* Moreover, we have that

$$\mathcal{E}'_\lambda(u)(v) = \int_M (\langle \nabla_g u, \nabla_g v \rangle + uv + e\phi_u uv) dv_g - \lambda \int_M \alpha(x) f(u) v dv_g. \quad (2.9)$$

In the sequel, let $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(\mathbf{H}_G^{x_0})$ and $(\boldsymbol{\alpha}^{x_0})$ are satisfied. The action of G on $H_g^1(M)$ is defined by

$$(\sigma u)(x) = u(\sigma^{-1}(x)) \text{ for all } \sigma \in G, u \in H_g^1(M), x \in M, \quad (2.10)$$

where $\sigma^{-1} : M \rightarrow M$ is the inverse of the isometry σ . Let

$$H_{g,G}^1(M) = \{u \in H_g^1(M) : \sigma u = u \text{ for all } \sigma \in G\}$$

be the subspace of G -invariant functions of $H_g^1(M)$ and $\mathcal{E}_{\lambda,G} : H_{g,G}^1(M) \rightarrow \mathbb{R}$ be the restriction of the energy functional \mathcal{E}_λ to $H_{g,G}^1(M)$. The following statement is crucial in our investigation:

STEP 3. *If $u_G \in H_{g,G}^1(M)$ is a critical point of $\mathcal{E}_{\lambda,G}$, then it is a critical point also for \mathcal{E}_λ and ϕ_{u_G} is G -invariant.*

Proof of Step 3. For the first part of the proof, we follow Kristály [18, Lemma 4.1]. Due to relation (2.10), the group G acts continuously on $H_g^1(M)$.

We claim that \mathcal{E}_λ is G -invariant. To prove this, let $u \in H_g^1(M)$ and $\sigma \in G$ be fixed. Since $\sigma : M \rightarrow M$ is an isometry on M , we have by (2.10) and the chain rule that $\nabla_g(\sigma u)(x) = D\sigma_{\sigma^{-1}(x)} \nabla_g u(\sigma^{-1}(x))$ for every $x \in M$, where $D\sigma_{\sigma^{-1}(x)} : T_{\sigma^{-1}(x)} M \rightarrow T_x M$ denotes the differential of σ at the point $\sigma^{-1}(x)$. The (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products; thus, by relation (2.10) and a change of variables $y = \sigma^{-1}(x)$ it

turns out that

$$\begin{aligned}
\|\sigma u\|_{H_g^1(M)}^2 &= \int_M (|\nabla_g(\sigma u)(x)|_x^2 + |(\sigma u)(x)|^2) dv_g(x) \\
&= \int_M (|\nabla_g u(\sigma^{-1}(x))|_{\sigma^{-1}(x)}^2 + |u(\sigma^{-1}(x))|^2) dv_g(x) \\
&= \int_M (|\nabla_g u(y)|_y^2 + |u(y)|^2) dv_g(y) \\
&= \|u\|_{H_g^1(M)}^2.
\end{aligned}$$

According to (α^{x_0}) , one has that $\alpha(x) = \alpha_0(d_g(x_0, x))$ for some function $\alpha_0 : [0, \infty) \rightarrow \mathbb{R}$. Since $\text{Fix}_M(G) = \{x_0\}$, we have for every $\sigma \in G$ and $x \in M$ that

$$\alpha(\sigma(x)) = \alpha_0(d_g(x_0, \sigma(x))) = \alpha_0(d_g(\sigma(x_0), \sigma(x))) = \alpha_0(d_g(x_0, x)) = \alpha(x).$$

Therefore,

$$\begin{aligned}
\mathcal{F}(\sigma u) &= \int_M \alpha(x) F((\sigma u)(x)) dv_g(x) = \int_M \alpha(x) F(u(\sigma^{-1}(x))) dv_g(x) = \int_M \alpha(y) F(u(y)) dv_g(y) \\
&= \mathcal{F}(u).
\end{aligned}$$

We now consider the Maxwell equation $-\Delta_g \phi_{\sigma u} + \phi_{\sigma u} = q(\sigma u)^2$ which reads pointwisely as $-\Delta_g \phi_{\sigma u}(y) + \phi_{\sigma u}(y) = qu(\sigma^{-1}(y))^2$, $y \in M$. After a change of variables one has $-\Delta_g \phi_{\sigma u}(\sigma(x)) + \phi_{\sigma u}(\sigma(x)) = qu(x)^2$, $x \in M$, which means by the uniqueness that $\phi_{\sigma u}(\sigma(x)) = \phi_u(x)$. Therefore,

$$\int_M \phi_{\sigma u}(x) (\sigma u)^2(x) dv_g(x) = \int_M \phi_u(\sigma^{-1}(x)) u^2(\sigma^{-1}(x)) dv_g(x) \stackrel{x=\sigma(y)}{=} \int_M \phi_u(y) u^2(y) dv_g(y),$$

which proves the G -invariance of $u \mapsto \int_M \phi_u u^2 dv_g$, thus the claim.

Since the fixed point set of $H_g^1(M)$ for G is precisely $H_{g,G}^1(M)$, the principle of symmetric criticality of Palais [22] shows that every critical point $u_G \in H_{g,G}^1(M)$ of the functional $\mathcal{E}_{\lambda,G}$ is also a critical point of \mathcal{E}_λ . Moreover, from the above uniqueness argument, for every $\sigma \in G$ and $x \in M$ we have $\phi_{u_G}(\sigma x) = \phi_{\sigma u_G}(\sigma x) = \phi_{u_G}(x)$, i.e., ϕ_{u_G} is G -invariant. \square

Summing up STEPS 1-3, we have the following implications: for an element $u \in H_{g,G}^1(M)$,

$$\mathcal{E}'_{\lambda,G}(u) = 0 \Rightarrow \mathcal{E}'_\lambda(u) = 0 \Leftrightarrow \mathcal{J}'_\lambda(u, \phi_u) = 0 \Leftrightarrow (u, \phi_u) \text{ is a weak solution of } (\mathcal{SM}_\lambda). \quad (2.11)$$

Consequently, in order to guarantee G -invariant weak solutions for (\mathcal{SM}_λ) , it is enough to produce critical points for the energy functional $\mathcal{E}_{\lambda,G} : H_{g,G}^1(M) \rightarrow \mathbb{R}$. While the embedding $H_g^1(M) \hookrightarrow L^p(M)$ is only continuous for every $p \in [2, 2^*]$, we adapt the main results from Skrzypczak and Tintarev [27] in order to regain some compactness by exploring the presence of group symmetries:

Proposition 2.1. [27, Theorem 1.3 & Proposition 3.1] *Let (M, g) be an n -dimensional homogeneous Hadamard manifold and G be a compact connected subgroup of $\text{Isom}_g(M)$ such that $\text{Fix}_M(G)$ is a singleton. Then $H_{g,G}^1(M)$ is compactly embedded into $L^p(M)$ for every $p \in (2, 2^*)$.*

3. PROOF OF THE MAIN RESULTS

3.1. Schrödinger-Maxwell systems of Poisson type. Consider the operator \mathcal{L} on $H_g^1(M)$ given by

$$\mathcal{L}(u) = -\Delta_g u + u + e\phi_u u.$$

The following comparison principle can be stated:

Lemma 3.1. *Let (M, g) be an n -dimensional Hadamard manifold ($3 \leq n \leq 6$), $u, v \in H_g^1(M)$.*

- (i) *If $\mathcal{L}(u) \leq \mathcal{L}(v)$ then $u \leq v$.*
- (ii) *If $0 \leq u \leq v$ then $\phi_u \leq \phi_v$.*

Proof. (i) Assume that $A = \{x \in M : u(x) > v(x)\}$ has a positive Riemannian measure. Then multiplying $\mathcal{L}(u) \leq \mathcal{L}(v)$ by $(u - v)_+$, an integration yields that

$$\int_A |\nabla_g u - \nabla_g v|^2 dv_g + \int_A (u - v)^2 dv_g + e \int_A (u\phi_u - v\phi_v)(u - v) dv_g \leq 0.$$

The latter inequality and relation (2.7) produce a contradiction.

(ii) Assume that $B = \{x \in M : \phi_u(x) > \phi_v(x)\}$ has a positive Riemannian measure. Multiplying the Maxwell-type equation $-\Delta_g(\phi_u - \phi_v) + \phi_u - \phi_v = q(u^2 - v^2)$ by $(\phi_u - \phi_v)_+$, we obtain that

$$\int_B |\nabla_g \phi_u - \nabla_g \phi_v|^2 dv_g + \int_B (\phi_u - \phi_v)^2 dv_g = q \int_B (u^2 - v^2)(\phi_u - \phi_v) dv_g \leq 0,$$

a contradiction. □

Proof of Theorem 1.1. Let $\lambda = 1$ and for simplicity, let $\mathcal{E} = \mathcal{E}_1$ be the energy functional from (2.8). First of all, the function $u \mapsto \frac{1}{2}\|u\|_{H_g^1(M)}^2$ is strictly convex on $H_g^1(M)$. Moreover, the linearity of $u \mapsto \mathcal{F}(u) = \int_M \alpha(x)u(x)dv_g(x)$ and property (2.6) imply that the energy functional \mathcal{E} is strictly convex on $H_g^1(M)$. Thus \mathcal{E} is sequentially weakly lower semicontinuous on $H_g^1(M)$, it is bounded from below and coercive. Now the basic result of the calculus of variations implies that \mathcal{E} has a unique (global) minimum point $u \in H_g^1(M)$, see Zeidler [34, Theorem 38.C and Proposition 38.15], which is also the unique critical point of \mathcal{E} , thus (u, ϕ_u) is the unique weak solution of (\mathcal{SM}) . Since $\alpha \geq 0$, Lemma 3.1 (i) implies that $u \geq 0$.

Assume the function α satisfies (α^{x_0}) for some $x_0 \in M$ and let $G \subset \text{Isom}_g(M)$ be such that $(H_G^{x_0})$ holds. Then we can repeat the above arguments for $\mathcal{E}_{1,G} = \mathcal{E}|_{H_{g,G}^1(M)}$ and $H_{g,G}^1(M)$ instead of \mathcal{E} and $H_g^1(M)$, respectively, obtaining by (2.11) that (u, ϕ_u) is a G -invariant weak solution for (\mathcal{SM}) . □

In the sequel we focus our attention to the system (\mathcal{R}) from §1; namely, we have

Lemma 3.2. *System (\mathcal{R}) has a unique, non-negative pair of solutions belonging to $C^\infty(0, \infty) \times C^\infty(0, \infty)$.*

Proof. Let $c \leq 0$ and $\alpha_0 \in L^2([0, \infty), \mathbf{s}_c(r)^{n-1}dr)$. Let us consider the Riemannian space form (M_c, g_c) with constant sectional curvature $c \leq 0$, i.e., (M_c, g_c) is either the Euclidean space $(\mathbb{R}^n, g_{\text{euc}})$ when $c = 0$, or the hyperbolic space $(\mathbb{H}^n, g_{\text{hyp}})$ with (scaled) sectional curvature $c < 0$. Let $x_0 \in M$ be fixed and $\alpha(x) = \alpha_0(d_{g_c}(x_0, x))$, $x \in M$. Due to the integrability

assumption on α_0 , we have that $\alpha \in L^2(M)$. Therefore, we are in the position to apply Theorem 1.1 on (M_c, g_c) (see examples from Remark 1.1) to the problem

$$\begin{cases} -\Delta_g u + u + eu\phi = \alpha(x) & \text{in } M_c, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M_c, \end{cases} \quad (\mathcal{SM}_c)$$

concluding that it has a unique, non-negative weak solution $(u_0, \phi_{u_0}) \in H_{g_c}^1(M_c) \times H_{g_c}^1(M_c)$, where u_0 is the unique global minimum point of the "one-variable" energy functional associated with problem (\mathcal{SM}_c) . Since α is radially symmetric in M_c , we may consider the group $G = SO(n)$ in the second part of Theorem 1.1 in order to prove that (u_0, ϕ_{u_0}) is $SO(n)$ -invariant, i.e., radially symmetric. In particular, we can represent these functions as $u_0(x) = h_1^c(d_{g_c}(x_0, x))$ and $\phi_0(x) = h_2^c(d_{g_c}(x_0, x))$ for some $h_i^c : [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2$. By using formula (2.2) and the Laplace comparison theorem for $\mathbf{K} = c$ it follows that the equations from (\mathcal{SM}_c) are transformed into the first two equations of (\mathcal{R}) while the second two relations in (\mathcal{R}) are nothing but the "radial" integrability conditions inherited from the fact that $(u_0, \phi_{u_0}) \in H_{g_c}^1(M_c) \times H_{g_c}^1(M_c)$. Thus, it turns out that problem (\mathcal{R}) has a non-negative pair of solutions (h_1^c, h_2^c) . Standard regularity results show that $(h_1^c, h_2^c) \in C^\infty(0, \infty) \times C^\infty(0, \infty)$. Finally, let us assume that (\mathcal{R}) has another non-negative pair of solutions $(\tilde{h}_1^c, \tilde{h}_2^c)$, distinct from (h_1^c, h_2^c) . Let $\tilde{u}_0(x) = \tilde{h}_1^c(d_{g_c}(x_0, x))$ and $\tilde{\phi}_0(x) = \tilde{h}_2^c(d_{g_c}(x_0, x))$. There are two cases: (a) if $h_1^c = \tilde{h}_1^c$ then $u_0 = \tilde{u}_0$ and by the uniqueness of solution for the Maxwell equation it follows that $\phi_0 = \tilde{\phi}_0$, i.e., $h_2^c = \tilde{h}_2^c$, a contradiction; (b) if $h_1^c \neq \tilde{h}_1^c$ then $u_0 \neq \tilde{u}_0$. But the latter relation is absurd since both elements u_0 and \tilde{u}_0 appear as unique global minima of the "one-variable" energy functional associated with (\mathcal{SM}_c) . \square

Proof of Theorem 1.2. "(ii) \Rightarrow (i)": it follows directly from Lemma 3.2.

"(i) \Rightarrow (ii)": Let $x_0 \in M$ be fixed and assume that the pair $(h_1^c(d_g(x_0, \cdot)), h_2^c(d_g(x_0, \cdot)))$ is the unique pointwise solution to (\mathcal{SM}) , i.e.,

$$\begin{cases} -\Delta_g h_1^c(d_g(x_0, x)) + h_1^c(d_g(x_0, x)) + eh_1^c(d_g(x_0, x))h_2^c(d_g(x_0, x)) = \alpha(d_g(x_0, x)), & x \in M, \\ -\Delta_g h_2^c(d_g(x_0, x)) + h_2^c(d_g(x_0, x)) = qh_1^c(d_g(x_0, x))^2, & x \in M. \end{cases}$$

By applying formula (2.2) to the second equation, we arrive to

$$-h_2^c(d_g(x_0, x))'' - h_2^c(d_g(x_0, x))' \Delta_g(d_g(x_0, x)) + h_2^c(d_g(x_0, x)) = qh_1^c(d_g(x_0, x))^2, \quad x \in M.$$

Subtracting the second equation of the system (\mathcal{R}) from the above one, we have that

$$h_2^c(d_g(x_0, x))' [\Delta_g(d_g(x_0, x)) - (n-1)\mathbf{ct}_c(d_g(x_0, x))] = 0, \quad x \in M. \quad (3.1)$$

Let us suppose that there exists a set $A \subset M$ of non-zero Riemannian measure such that $h_2^c(d_g(x_0, x))' = 0$ for every $x \in A$. By a continuity reason, there exists a non-degenerate interval $I \subset \mathbb{R}$ and a constant $c_0 \geq 0$ such that $h_2^c(t) = c_0$ for every $t \in I$. Coming back to the system (\mathcal{R}) , we observe that $h_1^c(t) = \sqrt{\frac{c_0}{q}}$ and $\alpha_0(t) = \sqrt{\frac{c_0}{q}}(1 + ec_0)$ for every $t \in I$. Therefore, $\alpha(x) = \alpha_0(d_g(x_0, x)) = \sqrt{\frac{c_0}{q}}(1 + ec_0)$ for every $x \in A$, which contradicts the assumption on α .

Consequently, by (3.1) we have $\Delta_g d_g(x_0, x) = (n-1)\mathbf{ct}_c(d_g(x_0, x))$ pointwisely on M . The latter relation can be equivalently transformed into

$$\Delta_g w_c(d_g(x_0, x)) = 1, \quad x \in M,$$

where

$$w_c(r) = \int_0^r \mathbf{s}_c(s)^{-n+1} \int_0^s \mathbf{s}_c(t)^{n-1} dt ds. \quad (3.2)$$

Let $0 < \tau$ be fixed arbitrarily. The unit outward normal vector to the forward geodesic sphere $S_g(x_0, \tau) = \partial B_g(x_0, \tau) = \{x \in M : d_g(x_0, x) = \tau\}$ at $x \in S_g(x_0, \tau)$ is given by $\mathbf{n} = \nabla_g d_g(x_0, x)$. Let us denote by $d\zeta_g(x)$ the canonical volume form on $S_g(x_0, \tau)$ induced by $dv_g(x)$. By Stoke's formula and $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ we have that

$$\begin{aligned} \text{Vol}_g(B_g(x_0, \tau)) &= \int_{B_g(x_0, \tau)} \Delta_g(w_c(d_g(x_0, x))) dv_g = \int_{B_g(x_0, \tau)} \text{div}(\nabla_g(w_c(d_g(x_0, x)))) dv_g \\ &= \int_{S_g(x_0, \tau)} \langle \mathbf{n}, w'_c(d_g(x_0, x)) \nabla_g d_g(x_0, x) \rangle dv_g \\ &= w'_c(\tau) \text{Area}_g(S_g(x_0, \tau)). \end{aligned}$$

Therefore,

$$\frac{\text{Area}_g(S_g(x_0, \tau))}{\text{Vol}_g(B_g(x_0, \tau))} = \frac{1}{w'_c(\tau)} = \frac{\mathbf{s}_c(\tau)^{n-1}}{\int_0^\tau \mathbf{s}_c(t)^{n-1} dt}.$$

Integrating the latter expression, it follows that

$$\frac{\text{Vol}_g(B_g(x_0, \tau))}{V_{c,n}(\tau)} = \lim_{s \rightarrow 0^+} \frac{\text{Vol}_g(B_g(x_0, s))}{V_{c,n}(s)} = 1. \quad (3.3)$$

In fact, the Bishop-Gromov volume comparison theorem implies that

$$\frac{\text{Vol}_g(B_g(x, \tau))}{V_{c,n}(\tau)} = 1 \text{ for all } x \in M, \tau > 0.$$

Now, the above equality implies that the sectional curvature is constant, $\mathbf{K} = c$, which concludes the proof. \square

3.2. Schrödinger-Maxwell systems involving sublinear terms at infinity. In this subsection we prove Theorem 1.3.

(i) Let $\lambda \geq 0$. If we choose $v = u$ in (1.3) we obtain that

$$\int_M (|\nabla_g u|^2 + u^2 + e\phi_u u^2) dv_g = \lambda \int_M \alpha(x) f(u) u dv_g.$$

Due to the assumptions $(f_1) - (f_3)$, the number $c_f = \max_{s>0} \frac{f(s)}{s}$ is well-defined and positive.

Thus, by (2.5) we have that

$$\|u\|_{H_g^1(M)}^2 \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \int_M u^2 dv_g \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \|u\|_{H_g^1(M)}^2.$$

Therefore, if $\lambda < c_f^{-1} \|\alpha\|_{L^\infty(M)}^{-1} := \tilde{\lambda}_0$, then the last inequality gives $u = 0$. By the Maxwell equation we also have that $\phi = 0$, which concludes the proof of (i).

(ii) The proof is divided into several steps.

CLAIM 1. *The energy functional \mathcal{E}_λ is coercive for every $\lambda \geq 0$. Indeed, due to (f_2) , we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(s)| \leq \varepsilon |s|^2$ for every $|s| > \delta$. Thus*

$$\mathcal{F}(u) = \int_{\{u>\delta\}} \alpha(x) F(u) dv_g + \int_{\{u \leq \delta\}} \alpha(x) F(u) dv_g \leq \varepsilon \|\alpha\|_{L^\infty(M)} \|u\|_{H_g^1(M)}^2 + \|\alpha\|_{L^1(M)} \max_{|s| \leq \delta} |F(s)|.$$

Therefore (see (2.8)),

$$\mathcal{E}_\lambda(u) \geq \left(\frac{1}{2} - \varepsilon \lambda \|\alpha\|_{L^\infty(M)} \right) \|u\|_{H_g^1(M)}^2 - \lambda \|\alpha\|_{L^1(M)} \sup_{|s| \leq \delta} |F(s)|.$$

In particular, if $0 < \varepsilon < (2\lambda \|\alpha\|_{L^\infty(M)})^{-1}$ then $\mathcal{E}_\lambda(u) \rightarrow \infty$ as $\|u\|_{H_g^1(M)} \rightarrow \infty$.

CLAIM 2. $\mathcal{E}_{\lambda,G}$ satisfies the Palais-Smale condition for every $\lambda \geq 0$. Let $\{u_j\}_j \subset H_{g,G}^1(M)$ be a Palais-Smale sequence, i.e., $\{\mathcal{E}_{\lambda,G}(u_j)\}$ is bounded and $\|(\mathcal{E}_{\lambda,G})'(u_j)\|_{H_{g,G}^1(M)^*} \rightarrow 0$ as $j \rightarrow \infty$. Since $\mathcal{E}_{\lambda,G}$ is coercive, the sequence $\{u_j\}_j$ is bounded in $H_{g,G}^1(M)$. Therefore, up to a subsequence, Proposition 2.1 implies that $\{u_j\}_j$ converges weakly in $H_{g,G}^1(M)$ and strongly in $L^p(M)$, $p \in (2, 2^*)$, to an element $u \in H_{g,G}^1(M)$. Note that

$$\begin{aligned} & \int_M |\nabla_g u_j - \nabla_g u|^2 dv_g + \int_M (u_j - u)^2 dv_g = \\ & (\mathcal{E}_{\lambda,G})'(u_j)(u_j - u) + (\mathcal{E}_{\lambda,G})'(u)(u - u_j) + \lambda \int_M \alpha(x)[f(u_j(x)) - f(u(x))](u_j - u) dv_g. \end{aligned}$$

Since $\|(\mathcal{E}_{\lambda,G})'(u_j)\|_{H_{g,G}^1(M)^*} \rightarrow 0$ and $u_j \rightharpoonup u$ in $H_{g,G}^1(M)$, the first two terms at the right hand side tend to 0. Let $p \in (2, 2^*)$. By the assumptions, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that $|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1}$ for every $s \in \mathbb{R}$. The latter relation, Hölder inequality and the fact that $u_j \rightarrow u$ in $L^p(M)$ imply that

$$\left| \int_M \alpha(x)[f(u_j) - f(u)](u_j - u) dv_g \right| \rightarrow 0,$$

as $j \rightarrow \infty$. Therefore, $\|u_j - u\|_{H_g^1(M)}^2 \rightarrow 0$ as $j \rightarrow \infty$.

CLAIM 3. $\mathcal{E}_{\lambda,G}$ is sequentially weakly lower semicontinuous for every $\lambda \geq 0$. First, since $\|\cdot\|_{H_g^1(M)}^2$ is convex, it is also sequentially weakly lower semicontinuous on $H_{g,G}^1(M)$. We shall prove that if $u_j \rightharpoonup u$ in $H_{g,G}^1(M)$, then $\int_M \phi_{u_j} u_j^2 dv_g \rightarrow \int_M \phi_u u^2 dv_g$. To see this, by Proposition 2.1 we have (up to a subsequence) that $\{u_j\}_j$ converges to u strongly in $L^p(M)$, $p \in (2, 2^*)$. Let us consider the Maxwell equations $-\Delta_g \phi_{u_j} + \phi_{u_j} = q u_j^2$ and $-\Delta_g \phi_u + \phi_u = q u^2$. Subtracting one from another and multiplying the expression by $(\phi_{u_j} - \phi_u)$, an integration and Hölder inequality yield that

$$\|\phi_{u_j} - \phi_u\|_{H_g^1(M)}^2 = q \int_M (u_j^2 - u^2)(\phi_{u_j} - \phi_u) dv_g \leq C \|u_j - u\|_{L^{\frac{4n}{n+2}}(M)} \|u_j + u\|_{H_g^1(M)} \|\phi_{u_j} - \phi_u\|_{H_g^1(M)},$$

for some $C > 0$. Since $\frac{4n}{n+2} < 2^*$ (note that $n \leq 5$), the first term of the right hand side tends to 0, thus we get that $\phi_{u_j} \rightarrow \phi_u$ in $H_{g,G}^1(M)$ as $j \rightarrow \infty$. Now, the desired limit follows from a Hölder inequality.

It remains to prove that \mathcal{F} is sequentially weakly continuous. To see this, let us suppose the contrary, i.e., let $\{u_j\} \subset H_{g,G}^1(M)$ be a sequence which converges weakly to $u \in H_{g,G}^1(M)$ and there exists $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 \leq |\mathcal{F}(u_j) - \mathcal{F}(u)|$ for every $j \in \mathbb{N}$. As before, $u_j \rightarrow u$ strongly in $L^p(M)$, $p \in (2, 2^*)$. By the mean value theorem one can see that for every $j \in \mathbb{N}$ there exists $0 < \theta_j < 1$ such that

$$0 < \varepsilon_0 \leq |\mathcal{F}(u_j) - \mathcal{F}(u)| \leq \int_M \alpha(x) |f(u + \theta_j(u_j - u))| |u_j - u| dv_g.$$

Now using assumptions (f_1) and (f_2) , the right hand side of the above estimate tends to 0, a contradiction. Thus, the energy functional $\mathcal{E}_{\lambda,G}$ is sequentially weakly lower semicontinuous.

CLAIM 4. (*First solution*) By using assumptions (f_1) and (f_2) , one has

$$\lim_{\mathcal{H}(u) \rightarrow 0} \frac{\mathcal{F}(u)}{\mathcal{H}(u)} = \lim_{\mathcal{H}(u) \rightarrow \infty} \frac{\mathcal{F}(u)}{\mathcal{H}(u)} = 0,$$

where $\mathcal{H}(u) = \frac{1}{2}\|u\|_{H_g^1(M)}^2 + \frac{e}{4} \int_M \phi_u u^2 dv_g$. Since $\alpha \in L^\infty(M)_+ \setminus \{0\}$, on account of (f_3) , one can guarantee the existence of a suitable truncation function $u_T \in H_{g,G}^1(M) \setminus \{0\}$ such that $\mathcal{F}(u_T) > 0$. Therefore, we may define

$$\lambda_0 = \inf_{\substack{u \in H_{g,G}^1(M) \setminus \{0\} \\ \mathcal{F}(u) > 0}} \frac{\mathcal{H}(u)}{\mathcal{F}(u)}.$$

The above limits imply that $0 < \lambda_0 < \infty$. By CLAIMS 1, 2 and 3, for every $\lambda > \lambda_0$, the functional $\mathcal{E}_{\lambda,G}$ is bounded from below, coercive and satisfies the Palais-Smale condition. If we fix $\lambda > \lambda_0$ one can choose a function $w \in H_{g,G}^1(M)$ such that $\mathcal{F}(w) > 0$ and $\lambda > \frac{\mathcal{H}(w)}{\mathcal{F}(w)} \geq \lambda_0$. In particular, $c_1 := \inf_{H_{g,G}^1(M)} \mathcal{E}_{\lambda,G} \leq \mathcal{E}_{\lambda,G}(w) = \mathcal{H}(w) - \lambda \mathcal{F}(w) < 0$. The latter inequality proves that the global minimum $u_{\lambda,G}^1 \in H_{g,G}^1(M)$ of $\mathcal{E}_{\lambda,G}$ on $H_{g,G}^1(M)$ has negative energy level. In particular, $(u_{\lambda,G}^1, \phi_{u_{\lambda,G}^1}) \in H_{g,G}^1(M) \times H_{g,G}^1(M)$ is a nontrivial weak solution to (\mathcal{SM}_λ) .

CLAIM 5. (*Second solution*) Let $q \in (2, 2^*)$ be fixed. By assumptions, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$0 \leq |f(s)| \leq \frac{\varepsilon}{\|\alpha\|_{L^\infty(M)}} |s| + C_\varepsilon |s|^{q-1} \text{ for all } s \in \mathbb{R}.$$

Then

$$\begin{aligned} 0 \leq |\mathcal{F}(u)| &\leq \int_M \alpha(x) |F(u(x))| dv_g \\ &\leq \int_M \alpha(x) \left(\frac{\varepsilon}{2\|\alpha\|_{L^\infty(M)}} u^2(x) + \frac{C_\varepsilon}{q} |u(x)|^q \right) dv_g \\ &\leq \frac{\varepsilon}{2} \|u\|_{H_g^1(M)}^2 + \frac{C_\varepsilon}{q} \|\alpha\|_{L^\infty(M)} \kappa_q^q \|u\|_{H_g^1(M)}^q, \end{aligned}$$

where κ_q is the embedding constant in $H_{g,G}^1(M) \hookrightarrow L^q(M)$. Thus,

$$\mathcal{E}_{\lambda,G}(u) \geq \frac{1}{2}(1 - \lambda\varepsilon) \|u\|_{H_g^1(M)}^2 - \frac{\lambda C_\varepsilon}{q} \|\alpha\|_\infty \kappa_q^q \|u\|_{H_g^1(M)}^q.$$

Bearing in mind that $q > 2$, for enough small $\rho > 0$ and $\varepsilon < \lambda^{-1}$ we have that

$$\inf_{\|u\|_{H_{g,G}^1(M)} = \rho} \mathcal{E}_{\lambda,G}(u) \geq \frac{1}{2}(1 - \varepsilon\lambda) \rho^2 - \frac{\lambda C_\varepsilon}{q} \|\alpha\|_{L^\infty(M)} \kappa_q^q \rho^{\frac{q}{2}} > 0.$$

A standard mountain pass argument (see [20, 32]) implies the existence of a critical point $u_{\lambda,G}^2 \in H_{g,G}^1(M)$ for $\mathcal{E}_{\lambda,G}$ with positive energy level. Thus $(u_{\lambda,G}^2, \phi_{u_{\lambda,G}^2}) \in H_{g,G}^1(M) \times H_{g,G}^1(M)$ is also a nontrivial weak solution to (\mathcal{SM}_λ) . Clearly, $u_{\lambda,G}^1 \neq u_{\lambda,G}^2$. \square

3.3. Schrödinger-Maxwell systems involving oscillatory nonlinearities. Before proving Theorem 1.4, we need an auxiliary result. Let us consider the system

$$\begin{cases} -\Delta_g u + u + eu\phi = \alpha(x)\tilde{f}(u) & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \quad (\widetilde{\mathcal{SM}}),$$

where the following assumptions hold:

- (\tilde{f}_1) $\tilde{f} : [0, \infty) \rightarrow \mathbb{R}$ is a bounded function such that $f(0) = 0$;
- (\tilde{f}_2) there are $0 < a \leq b$ such that $\tilde{f}(s) \leq 0$ for all $s \in [a, b]$.

Let $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(H_G^{x_0})$ and (α^{x_0}) are satisfied.

Let $\tilde{\mathcal{E}}$ be the "one-variable" energy functional associated with system $(\widetilde{\mathcal{SM}})$, and $\tilde{\mathcal{E}}_G$ be the restriction of $\tilde{\mathcal{E}}$ to the set $H_{g,G}^1(M)$. It is clear that $\tilde{\mathcal{E}}$ is well defined. Consider the number $b \in \mathbb{R}$ from (\tilde{f}_2) ; for further use, we introduce the sets

$$W^b = \{u \in H_g^1(M) : \|u\|_{L^\infty(M)} \leq b\} \quad \text{and} \quad W_G^b = W^b \cap H_{g,G}^1(M).$$

Proposition 3.1. *Let (M, g) be an n -dimensional homogeneous Hadamard manifold ($3 \leq n \leq 5$), $x_0 \in M$ be fixed, and $G \subset \text{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(H_G^{x_0})$ and (α^{x_0}) are satisfied. If $\tilde{f} : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function satisfying (\tilde{f}_1) and (\tilde{f}_2) then*

- (i) *the infimum of $\tilde{\mathcal{E}}_G$ on W_G^b is attained at an element $u_G \in W_G^b$;*
- (ii) *$u_G(x) \in [0, a]$ a.e. $x \in M$;*
- (iii) *(u_G, ϕ_{u_G}) is a weak solution to system $(\widetilde{\mathcal{SM}})$.*

Proof. (i) By using the same method as in CLAIM 3 of the proof of Theorem 1.3, the functional $\tilde{\mathcal{E}}_G$ is sequentially weakly lower semicontinuous on $H_{g,G}^1(M)$. Moreover, $\tilde{\mathcal{E}}_G$ is bounded from below. The set W_G^b is convex and closed in $H_{g,G}^1(M)$, thus weakly closed. Therefore, the claim directly follows; let $u_G \in W_G^b$ be the infimum of $\tilde{\mathcal{E}}_G$ on W_G^b .

(ii) Let $A = \{x \in M : u_G(x) \notin [0, a]\}$ and suppose that the Riemannian measure of A is positive. We consider the function $\gamma(s) = \min(s_+, a)$ and set $w = \gamma \circ u_G$. Since γ is Lipschitz continuous, then $w \in H_g^1(M)$ (see Hebey, [14, Proposition 2.5, page 24]). We claim that $w \in H_{g,G}^1(M)$. Indeed, for every $x \in M$ and $\sigma \in G$,

$$\sigma w(x) = w(\sigma^{-1}(x)) = (\gamma \circ u_G)(\sigma^{-1}(x)) = \gamma(u_G(\sigma^{-1}(x))) = \gamma(u_G(x)) = w(x).$$

By construction, we clearly have that $w \in W_G^b$. Let

$$A_1 = \{x \in A : u_G(x) < 0\} \text{ and } A_2 = \{x \in A : u_G(x) > a\}.$$

Thus $A = A_1 \cup A_2$, and from the construction we have that $w(x) = u_G(x)$ for all $x \in M \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = a$ for all $x \in A_2$. Now we have that

$$\begin{aligned} \tilde{\mathcal{E}}_G(w) - \tilde{\mathcal{E}}_G(u_G) &= -\frac{1}{2} \int_A |\nabla_g u_G|^2 dv_g + \frac{1}{2} \int_A (w^2 - u_G^2) dv_g + \frac{e}{4} \int_A (\phi_w w^2 - \phi_{u_G} u_G^2) dv_g \\ &\quad - \int_A \alpha(x) (\tilde{F}(w) - \tilde{F}(u_G)) dv_g. \end{aligned}$$

Note that

$$\int_A (w^2 - u_G^2) dv_g = - \int_{A_1} u_G^2 dv_g + \int_{A_2} (a^2 - u_G^2) dv_g \leq 0.$$

It is also clear that $\int_{A_1} \alpha(x)(\tilde{F}(w) - \tilde{F}(u_G))dv_g = 0$, and due to the mean value theorem and (\tilde{f}_2) we have that $\int_{A_2} \alpha(x)(\tilde{F}(w) - \tilde{F}(u_G))dv_g \geq 0$. Furthermore,

$$\int_A (\phi_w w^2 - \phi_{u_G} u_G^2)dv_g = - \int_{A_1} \phi_{u_G} u_G^2 dv_g + \int_{A_2} (\phi_w w^2 - \phi_{u_G} u_G^2)dv_g,$$

thus due to Lemma 3.1 (ii), since $0 \leq w \leq u_G$, we have that $\int_{A_2} (\phi_w w^2 - \phi_{u_G} u_G^2)dv_g \leq 0$.

Combining the above estimates, we have $\widetilde{\mathcal{E}}_G(w) - \widetilde{\mathcal{E}}_G(u_G) \leq 0$.

On the other hand, since $w \in W_G^b$ then $\mathcal{E}_G(w) \geq \mathcal{E}_G(u_G) = \inf_{W_G^b} \widetilde{\mathcal{E}}_G$, thus we necessarily have that

$$\int_{A_1} u_G^2 dv_g = \int_{A_2} (a^2 - u_G^2)dv_g = 0,$$

which implies that the Riemannian measure of A should be zero, a contradiction.

(iii) The proof is divided into two steps:

CLAIM 1. $\widetilde{\mathcal{E}}'(u_G)(w - u_G) \geq 0$ for all $w \in W^b$. It is clear that the set W^b is closed and convex in $H_g^1(M)$. Let χ_{W^b} be the indicator function of the set W^b , i.e., $\chi_{W^b}(u) = 0$ if $u \in W^b$, and $\chi_{W^b}(u) = +\infty$ otherwise. Let us consider the Szulkin-type functional $\mathcal{K} : H_g^1(M) \rightarrow \mathbb{R} \cup \{+\infty\}$ given by $\mathcal{K} = \widetilde{\mathcal{E}} + \chi_{W^b}$. On account of the definition of the set W_G^b , the restriction of χ_{W^b} to $H_{g,G}^1(M)$ is precisely the indicator function $\chi_{W_G^b}$ of the set W_G^b . By (i), since u_G is a local minimum point of $\widetilde{\mathcal{E}}_G$ relative to the set W_G^b , it is also a local minimum point of the Szulkin-type functional $\mathcal{K}_G = \widetilde{\mathcal{E}}_G + \chi_{W_G^b}$ on $H_{g,G}^1(M)$. In particular, u_G is a critical point of \mathcal{K}_G in the sense of Szulkin [28], i.e.,

$$0 \in \widetilde{\mathcal{E}}'_G(u_G) + \partial \chi_{W_G^b}(u_G) \text{ in } (H_{g,G}^1(M))^*,$$

where ∂ stands for the subdifferential in the sense of convex analysis. By exploring the compactness of the group G , we may apply the principle of symmetric criticality for Szulkin-type functionals, see Kobayashi and Ôtani [17, Theorem 3.16], obtaining that

$$0 \in \widetilde{\mathcal{E}}'(u_G) + \partial \chi_{W^b}(u_G) \text{ in } (H_g^1(M))^*.$$

Consequently, we have for every $w \in W^b$ that

$$0 \leq \widetilde{\mathcal{E}}'(u_G)(w - u_G) + \chi_{W^b}(w) - \chi_{W^b}(u_G),$$

which proves the claim.

CLAIM 2. (u_G, ϕ_{u_G}) is a weak solution to the system $(\widetilde{\mathcal{SM}})$. By assumption (\tilde{f}_1) it is clear that $C_m = \sup_{s \in \mathbb{R}} |\tilde{f}(s)| < \infty$. The previous step and (2.9) imply that for all $w \in W^b$,

$$\begin{aligned} 0 &\leq \int_M \langle \nabla_g u_G, \nabla_g (w - u_G) \rangle dv_g + \int_M u_G (w - u_G) dv_g \\ &\quad + e \int_M u_G \phi_{u_G} (w - u_G) dv_g - \int_M \alpha(x) \tilde{f}(u_G) (w - u_G) dv_g. \end{aligned}$$

Let us define the following function

$$\zeta(s) = \begin{cases} -b, & s < -b, \\ s, & -b \leq s < b, \\ b, & b \leq s. \end{cases}$$

Since ζ is Lipschitz continuous and $\zeta(0) = 0$, then for fixed $\varepsilon > 0$ and $v \in H_g^1(M)$ the function $w_\zeta = \zeta \circ (u_G + \varepsilon v)$ belongs to $H_g^1(M)$, see Hebey [14, Proposition 2.5, page 24]. By construction, $w_\zeta \in W^b$.

Let us denote by $B_1 = \{x \in M : u_G + \varepsilon v < -b\}$, $B_2 = \{x \in M : -b \leq u_G + \varepsilon v < b\}$ and $B_3 = \{x \in M : u_G + \varepsilon v \geq b\}$. Choosing $w = w_\zeta$ in the above inequality we have that

$$0 \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= - \int_{B_1} |\nabla_g u_G|^2 dv_g + \varepsilon \int_{B_2} \langle \nabla_g u_G, \nabla_g v \rangle dv_g - \int_{B_3} |\nabla_g u_G|^2 dv_g, \\ I_2 &= - \int_{B_1} u_G(b + u_G) dv_g + \varepsilon \int_{B_2} u_G v dv_g + \int_{B_3} (b - u_G) dv_g, \\ I_3 &= -e \int_{B_1} u_G \phi_{u_G}(b + u_G) dv_g + \varepsilon e \int_{B_2} u_G \phi_{u_G} v dv_g + e \int_{B_3} u_G \phi_{u_G}(b - u_G) dv_g, \end{aligned}$$

and

$$I_4 = - \int_{B_1} \alpha(x) \tilde{f}(u_G)(-b - u_G) dv_g - \varepsilon \int_{B_2} \alpha(x) \tilde{f}(u_G) v dv_g - \int_{B_3} \alpha(x) \tilde{f}(u_G)(b - u_G) dv_g.$$

After a rearrangement we obtain that

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 &= \varepsilon \int_M \langle \nabla_g u_G, \nabla_g v \rangle dv_g + \varepsilon \int_M u_G v dv_g + \varepsilon e \int_M u_G \phi_{u_G} v dv_g - \varepsilon \int_M \alpha(x) f(u_G) v dv_g \\ &\quad - \varepsilon \int_{B_1} \langle \nabla_g u_G, \nabla_g v \rangle dv_g - \varepsilon \int_{B_3} \langle \nabla_g u_G, \nabla_g v \rangle dv_g - \int_{B_1} |\nabla_g u_G|^2 dv_g \\ &\quad - \int_{B_3} |\nabla_g u_G|^2 dv_g + \int_{B_1} (b + u_G + \varepsilon v) \left(\alpha(x) \tilde{f}(u_G) - u_G - e u_G \phi_{u_G} \right) dv_g \\ &\quad + \int_{B_3} (-b + u_G + \varepsilon v) \left(\alpha(x) \tilde{f}(u_G) - u_G - e u_G \phi_{u_G} \right) dv_g. \end{aligned}$$

Note that

$$\int_{B_1} (b + u_G + \varepsilon v) \left(\alpha(x) \tilde{f}(u_G) - u_G - e u_G \phi_{u_G} \right) dv_g \leq -\varepsilon \int_{B_1} (C_{\mathfrak{m}} \alpha(x) + u_G + e u_G \phi_{u_G}) v dv_g,$$

and

$$\int_{B_3} (-b + u_G + \varepsilon v) \left(\alpha(x) \tilde{f}(u_G) - u_G - e u_G \phi_{u_G} \right) dv_g \leq \varepsilon C_{\mathfrak{m}} \int_{B_3} \alpha(x) v dv_g.$$

Now, using the above estimates and dividing by $\varepsilon > 0$, we have that

$$\begin{aligned} 0 &\leq \int_M \langle \nabla_g u_G, \nabla_g v \rangle dv_g + \int_M u_G v dv_g + e \int_M u_G \phi_{u_G} v dv_g - \int_M \alpha(x) \tilde{f}(u_G) v dv_g \\ &\quad - \int_{B_1} (\langle \nabla_g u_G, \nabla_g v \rangle + C_{\mathfrak{m}} \alpha(x) v + u_G v + e u_G \phi_{u_G} v) dv_g - \int_{B_3} (\langle \nabla_g u_G, \nabla_g v \rangle - C_{\mathfrak{m}} \alpha(x) v) dv_g. \end{aligned}$$

Taking into account that the Riemannian measures for both sets B_1 and B_3 tend to zero as $\varepsilon \rightarrow 0$, we get that

$$0 \leq \int_M \langle \nabla_g u_G, \nabla_g v \rangle dv_g + \int_M u_G v dv_g + e \int_M u_G \phi_{u_G} v dv_g - \int_M \alpha(x) \tilde{f}(u_G) v dv_g.$$

Replacing v by $(-v)$, it yields

$$0 = \int_M \langle \nabla_g u_G, \nabla_g v \rangle dv_g + \int_M u_G v dv_g + e \int_M u_G \phi_{u_G} v dv_g - \int_M \alpha(x) \tilde{f}(u_G) v dv_g,$$

i.e., $\tilde{\mathcal{E}}'(u_G) = 0$. Thus (u_G, ϕ_{u_G}) is a G -invariant weak solution to $(\widetilde{\mathcal{SM}})$. \square

Let $s > 0$, $0 < r < \rho$ and $A_{x_0}[r, \rho] = B_g(x_0, \rho + r) \setminus B_g(x_0, \rho - r)$ be an annulus-type domain. For further use, we define the function $w_s : M \rightarrow \mathbb{R}$ by

$$w_s(x) = \begin{cases} 0, & x \in M \setminus A_{x_0}[r, \rho], \\ s, & x \in A_{x_0}[r/2, \rho], \\ \frac{2s}{r}(r - |d_g(x_0, x) - \rho|), & x \in A_{x_0}[r, \rho] \setminus A_{x_0}[r/2, \rho]. \end{cases}$$

Note that $(H_G^{x_0})$ implies $w_s \in H_{g,G}^1(M)$.

Proof of Theorem 1.4. Due to (f_0^2) and the continuity of f one can fix two sequences $\{\theta_j\}_j, \{\eta_j\}_j$ such that $\lim_{j \rightarrow +\infty} \theta_j = \lim_{j \rightarrow +\infty} \eta_j = 0$, and for every $j \in \mathbb{N}$,

$$0 < \theta_{j+1} < \eta_j < s_j < \theta_j < 1; \quad (3.4)$$

$$f(s) \leq 0 \text{ for every } s \in [\eta_j, \theta_j]. \quad (3.5)$$

Let us introduce the auxiliary function $f_j(s) = f(\min(s, \theta_j))$. Since $f(0) = 0$ (by (f_0^1) and (f_0^2)), then $f_j(0) = 0$ and we may extend continuously the function f_j to the whole real line by $f_j(s) = 0$ if $s \leq 0$. For every $s \in \mathbb{R}$ and $j \in \mathbb{N}$, we define $F_j(s) = \int_0^s f_j(t) dt$. It is clear that f_j satisfies the assumptions (\tilde{f}_1) and (\tilde{f}_2) . Thus, applying Proposition 3.1 to the function f_j , $j \in \mathbb{N}$, the system

$$\begin{cases} -\Delta_g u + u + eu\phi = \alpha(x)f_j(u) & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \quad (3.6)$$

has a G -invariant weak solution $(u_j^0, \phi_{u_j^0}) \in H_{g,G}^1(M) \times H_{g,G}^1(M)$ such that

$$u_j^0 \in [0, \eta_j] \text{ a.e. } x \in M; \quad (3.7)$$

$$u_j^0 \text{ is the infimum of the functional } \mathcal{E}_j \text{ on the set } W_G^{\theta_j}, \quad (3.8)$$

where

$$\mathcal{E}_j(u) = \frac{1}{2} \|u\|_{H_g^1(M)}^2 + \frac{e}{4} \int_M \phi_u u^2 dv_g - \int_M \alpha(x) F_j(u) dv_g.$$

By (3.7), $(u_j^0, \phi_{u_j^0}) \in H_{g,G}^1(M) \times H_{g,G}^1(M)$ is also a weak solution to the initial system (\mathcal{SM}) .

It remains to prove the existence of infinitely many distinct elements in the sequence $\{(u_j^0, \phi_{u_j^0})\}_j$. First, due to (α^{x_0}) , there exist $0 < r < \rho$ such that $\text{essinf}_{A_{x_0}[r, \rho]} \alpha > 0$. For simplicity, let $D = A_{x_0}[r, \rho]$ and $K = A_{x_0}[r/2, \rho]$. By (f_0^1) there exist $l_0 > 0$ and $\delta \in (0, \theta_1)$ such that

$$F(s) \geq -l_0 s^2 \text{ for every } s \in (0, \delta). \quad (3.9)$$

Again, (f_0^1) implies the existence of a non-increasing sequence $\{\tilde{s}_j\}_j \subset (0, \delta)$ such that $\tilde{s}_j \leq \eta_j$ and

$$F(\tilde{s}_j) > L_0 \tilde{s}_j^2 \text{ for all } j \in \mathbb{N}, \quad (3.10)$$

where $L_0 > 0$ is enough large, e.g.,

$$L_0 \text{essinf}_K \alpha > \frac{1}{2} \left(1 + \frac{4}{r^2} \right) \text{Vol}_g(D) + \frac{e}{4} \|\phi_\delta\|_{L^1(D)} + l_0 \|\alpha\|_{L^1(M)}. \quad (3.11)$$

Note that

$$\mathcal{E}_j(w_{\tilde{s}_j}) = \frac{1}{2} \|w_{\tilde{s}_j}\|_{H_g^1(M)}^2 + \frac{e}{4} I_j - J_j,$$

where

$$I_j = \int_D \phi_{w_{\tilde{s}_j}} w_{\tilde{s}_j}^2 dv_g \text{ and } J_j = \int_D \alpha(x) F_j(w_{\tilde{s}_j}) dv_g.$$

By Lemma 3.1 (ii) we have

$$I_j \leq \tilde{s}_j^2 \|\phi_\delta\|_{L^1(D)}, \quad j \in \mathbb{N}.$$

Moreover, by (3.9) and (3.10) we have that

$$J_j \geq L_0 \tilde{s}_j^2 \text{essinf}_K \alpha - l_0 \tilde{s}_j^2 \|\alpha\|_{L^1(M)}, \quad j \in \mathbb{N}.$$

Therefore,

$$\mathcal{E}_j(w_{\tilde{s}_j}) \leq \tilde{s}_j^2 \left(\frac{1}{2} \left(1 + \frac{4}{r^2} \right) \text{Vol}_g(D) + \frac{e}{4} \|\phi_\delta\|_{L^1(D)} + l_0 \|\alpha\|_{L^1(M)} - L_0 \text{essinf}_K \alpha \right).$$

Thus, in one hand, by (3.11) we have

$$\mathcal{E}_j(u_j^0) = \inf_{W_G^{\theta_j}} \mathcal{E}_j \leq \mathcal{E}_j(w_{\tilde{s}_j}) < 0. \quad (3.12)$$

On the other hand, by (3.4) and (3.7) we clearly have

$$\mathcal{E}_j(u_j^0) \geq - \int_M \alpha(x) F_j(u_j^0) dv_g = - \int_M \alpha(x) F(u_j^0) dv_g \geq - \|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)| \eta_j, \quad j \in \mathbb{N}.$$

Combining the latter relations, it yields that $\lim_{j \rightarrow +\infty} \mathcal{E}_j(u_j^0) = 0$. Since $\mathcal{E}_j(u_j^0) = \mathcal{E}_1(u_j^0)$ for all $j \in \mathbb{N}$, we obtain that the sequence $\{u_j^0\}_j$ contains infinitely many distinct elements. In particular, by (3.12) we have that $\frac{1}{2} \|u_j^0\|_{H_g^1(M)}^2 \leq \|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)| \eta_j$, which implies that $\lim_{j \rightarrow \infty} \|u_j^0\|_{H_g^1(M)} = 0$. Recalling (2.5), we also have $\lim_{j \rightarrow \infty} \|\phi_{u_j^0}\|_{H_g^1(M)} = 0$, which concludes the proof. \square

Remark 3.1. Using Proposition 3.1 (i) and $\lim_{j \rightarrow \infty} \eta_j = 0$, it follows that $\lim_{j \rightarrow \infty} \|u_j^0\|_{L^\infty(M)} = 0$.

ACKNOWLEDGEMENT

The authors were supported by the grant of the Romanian National Authority for Scientific Research, "Symmetries in elliptic problems: Euclidean and non-Euclidean techniques", CNCS-UEFISCDI, project no. PN-II-ID-PCE-2011-3-0241. A. Kristály is also supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

REFERENCES

- [1] A. AMBROSETTI, D. RUIZ, *Multiple bound states for the Schrödinger-Poisson problem*. Commun. Contemp. Math. **10** (2008), no. 3, 391–404.
- [2] A. AZZOLLINI, *Concentration and compactness in nonlinear Schrödinger-Poisson system with a general nonlinearity*. J. Differential Equations **249** (7) (2010) 1746–1763.
- [3] A. AZZOLLINI, P. D’AVENIA, A. POMPONIO, *On the Schrödinger-Maxwell equations under the effect of a general nonlinear term*. Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2) (2010) 779–791.
- [4] V. BENCI, D. FORTUNATO, *An eigenvalue problem for the Schrödinger-Maxwell equations*. Top. Meth. Nonlinear Anal. **11** (1998) 283–293.
- [5] H. BREZIS, *Functional Analysis, Sobolev spaces and Partial Differential Equations*. Springer, 2010.
- [6] M. R. BRIDSON, A. HAEFLIGER, *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.
- [7] G. CERAMI, G. VAIRA, *Positive solutions for some non-autonomous Schrödinger-Poisson systems*. J. Differential Equations **248** (2010) 521–543.
- [8] T. D’APRILE, D. MUGNAI, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*. Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 893–906.
- [9] T. D’APRILE, D. MUGNAI, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*. Adv. Nonlinear Stud. **4** (2004) 307–322.
- [10] P. D’AVENIA, *Non-radially symmetric solution of the nonlinear Schrödinger equation coupled with Maxwell equations*. Adv. Nonlinear Stud. **2** (2002), 177–192.
- [11] O. DRUET, E. HEBEY, *Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces*. Commun. Contemp. Math. **12** (2010), no. 5, 831–869.
- [12] M. GHIMENTI, A. M. MICHELETTI, *Number and profile of low energy solutions for singularly perturbed Klein-Gordon-Maxwell systems on a Riemannian manifold*. J. Differential Equations **256** (2014), no. 7, 2502–2525.
- [13] M. GHIMENTI, A. M. MICHELETTI, *Low energy solutions for singularly perturbed coupled nonlinear systems on a Riemannian manifold with boundary*. Nonlinear Anal. **119** (2015), 315–329.
- [14] E. HEBEY, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes in Mathematics, New York, 1999.
- [15] E. HEBEY, J. WEI, *Schrödinger-Poisson systems in the 3-sphere*. Calc. Var. Partial Differential Equations **47** (2013), no. 1-2, 25–54.
- [16] D. HOFFMAN, J. SPRUCK, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*. Comm. Pure Appl. Math. **27** (1974), 715–727.
- [17] J. KOBAYASHI AND M. ÔTANI, *The principle of symmetric criticality for non-differentiable mappings*. J. Funct. Anal., **214** (2004), 428–449.
- [18] A. KRISTÁLY, *Geometric aspects of Moser-Trudinger inequalities on complete non-compact Riemannian manifolds with applications*, preprint, arXiv:1502.01439.
- [19] A. KRISTÁLY, G. MOROȘANU, *New competition phenomena in Dirichlet problems*. J. Math. Pures Appl. **94** (2010) 555–570.
- [20] A. KRISTÁLY, V. RĂDULESCU, CS. VARGA, *Variational principles in mathematical physics, geometry and economics*, Cambridge University Press, 2010.
- [21] A. KRISTÁLY, D. REPOVS, *On the Schrödinger-Maxwell system involving sublinear terms*. Nonlinear Analysis Real World Applications, **13**(1) (2012), 213–223.
- [22] R. PALAIS, *The principle of symmetric criticality*. Comm. Math. Phys. **69** (1979), no. 1, 19–30.
- [23] B. RICCI, *A further three critical points theorem*. Nonlinear Anal. **71** (2009), 4151–4157.
- [24] D. RUIZ, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*. J. Funct. Anal. **237** (2006) 655–674.
- [25] Z. SHEN, *Volume comparison and its applications in Riemann-Finsler geometry*. Adv. Math. **128** (1997), no. 2, 306–328.
- [26] J. SUN, H. CHEN, J. NIETO, *On ground state solutions for some non-autonomous Schrödinger-Poisson systems*. J. Differential Equations **252** (2012), no. 5, 3365–3380.
- [27] L. SKRZYPCZAK, C. TINTAREV, *A geometric criterion for compactness of invariant subspaces*. Arch. Math. **101** (2013), 259–268.

- [28] A. SZULKIN, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*. Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 2, 77–109.
- [29] P.-D. THIZY, *Blow-up for Schrödinger-Poisson critical systems in dimensions 4 and 5*. Calc. Var., 2016, in press. DOI: 10.1007/s00526-016-0959-x.
- [30] P.-D. THIZY, *Schrödinger-Poisson systems in 4-dimensional closed manifolds*. Discrete Contin. Dyn. Syst. **36** (2016), no. 4, 2257–2284.
- [31] Z. WANG, H.-S. ZHOU, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in \mathbb{R}^3* . Discrete Contin. Dyn. Syst. **18** (2007) 809–816.
- [32] M. WILLEM, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications **24**, Birkhäuser, Boston, 1996.
- [33] B. Y. WU, Y. L. XIN, *Comparison theorems in Finsler geometry and their applications*. Math. Ann. **337** (2007), no. 1, 177–196.
- [34] E. ZEIDLER, *Nonlinear functional analysis and its applications. III. Variational methods and optimization*, Springer-Verlag, New York, 1985.
- [35] L. ZHAO, H. LIU, F. ZHAO, *Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential*. J. Differential Equations **255** (2013), no. 1, 1–23.